Online Vertex-Weighted Bipartite Matching and Single-bid Budgeted Allocations

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Abstract

We study the following vertex-weighted online bipartite matching problem: $G(U, V, E)$ is a bipartite graph. The vertices in U have weights and are known ahead of time, while the vertices in V arrive online in an arbitrary order and have to be matched upon arrival. The goal is to maximize the sum of weights of the matched vertices in U. When all the weights are equal, this reduces to the classic *online bipartite matching* problem for which Karp, Vazirani and Vazirani gave an optimal $\left(1 - \frac{1}{e}\right)$-competitive algorithm in their seminal work [KVV90].

Our main result is an optimal $\left(1 - \frac{1}{e}\right)$-competitive randomized algorithm for general vertex weights. We use *random perturbations* of weights by appropriately chosen multiplicative factors. Our solution constitutes the first known generalization of the algorithm in [KVV90] in this model and provides new insights into the role of randomization in online allocation problems. It also effectively solves the problem of *online budgeted allocations* [MSVV05] in the case when an agent makes the same bid for any desired item, even if the bid is comparable to his budget - complementing the results of [MSVV05, BJN07] which apply when the bids are much smaller than the budgets.

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1 Introduction

Online bipartite matching is a fundamental problem with numerous applications such as matching candidates to jobs, ads to advertisers, or boys to girls. A canonical result in online bipartite matching is due to Karp, Vazirani and Vazirani [KVV90], who gave an optimal online algorithm for the unweighted case to maximize the *size* of the matching. In their model, we are given a bipartite graph $G(U, V, E)$. The vertices in U are known ahead of time, while the vertices in V arrive one at a time online in an arbitrary order. When a vertex in V arrives, the edges incident to it are revealed and it can be matched to a neighboring vertex in U that has not already been matched. A match once made cannot be revoked. The goal is to maximize the number of matched vertices.

However, in many real world scenarios, the value received from matching a vertex might be different for different vertices: (1) Advertisers in online display ad-campaigns are willing to pay a fixed amount every time their graphic ad is shown on a website. By specifying their targeting criteria, they can choose the set of websites they are interested in. Each impression of an ad can be thought of as matching the impression to the advertiser, collecting revenue equal to the advertiser's bid. (2) Consider the sale of an inventory of items such as cars. Buyers arrive in an online manner looking to purchase one out of a specified set of items they are interested in. The sale of an item generates revenue equal to the price of the item. The goal in both these cases is to maximize the total revenue. With this background, we consider the following problem:

Online vertex-weighted bipartite matching: The input instance is a bipartite graph $G(U, V, E, \{b_u\}_{u \in U})$, with the vertices in U and their weights b_u known ahead of time. Vertices in V arrive one at a time, online, revealing their incident edges. An arriving vertex can be matched to an unmatched neighbor upon arrival. Matches once made cannot be revoked later and a vertex left unmatched upon arrival cannot be matched later. The goal is to maximize the sum of the weights of the matched vertices in U.

Connection to the online budgeted allocation problem: Apart from being a natural generalization of the online bipartite matching problem, our vertex-weighted matching problem is closely related to an important class of online problems. Mehta *et al* [MSVV05] considered the following online version of maximum budgeted allocation problem [GKP01, LLN01] to model sponsored search auctions: We have n agents and m items. Each agent i specifies a monetary budget B_i and a bid b_{ij} for each item j. Items arrive online, and must be immediately allocated to an agent. If a set S of items is allocated to agent i, then the agent pays the minimum of B_i and $\sum_{j \in S} b_{ij}$. The objective is to maximize the total revenue of the algorithm. An important and unsolved restricted case of this problem is when all the non-zero bids of an agent are equal, *i.e.* $b_{ij} = b_i$ or 0 for all j. This case reduces to our vertex-weighted matching problem (For a proof, refer to Appendix A).

For the general online budgeted allocation problem, no factor better than $\frac{1}{2}$ (achieved by a simple deterministic greedy algorithm [LLN01]) is yet known. The best known lower bound stands at $1 - \frac{1}{e}$ due to the hardness result in [KVV90] for the case when all bids and budgets are equal to 1 - which is equivalent to the unweighted online matching problem. The *small bids* case - where $b_{ij} \ll B_i$ for all i and j - was solved by [MSVV05, BJN07] achieving the optimal $1 - \frac{1}{e}$ deterministic competitive ratio. It was believed that handling *large bids* requires the use of randomization, as in [KVV90]. In particular, many attempts [KV07, BM08, GM08] had been made to simplify the analysis of the randomized algorithm in [KVV90], but no generalization had been achieved.

Our solution to the vertex-weighted matching problem is a significant step in this direction. Our algorithm generalizes that of [KVV90] and provides new insights into the role of randomization in these solutions, as outlined in Section 1.1. Finally, our algorithm has interesting connections to the solution of [MSVV05] for the *small bids* case - despite the fact that the vertex-weighted matching problem is neither harder nor easier than the *small bids* case. This strongly suggests a possible unified approach to the unrestricted online budgeted allocation problem. See Section 1.2 for details.

1.1 Overview of the Result

Solution to the unweighted case: To describe our result, it is instructive to start at the unweighted case ($b_u = 1$ for all $u \in U$) and study its solution by [KVV90]. Two natural approaches that match each arriving $v \in V$ to the an unmatched neighbor in U chosen (a) arbitrarily and (b) randomly, both fail to achieve competitive ratio better than $\frac{1}{2}$. Their solution is an elegant randomized algorithm called RANKING that works as follows: it begins by picking a *uniformly random permutation* of the vertices in U (called the "ranking" of the vertices). Then, as a vertex in V arrives, it is matched to the highest-ranked unmatched neighbor. Surprisingly, this idea of using correlated randomness for all the arriving vertices achieves the optimal competitive ratio of $1-\frac{1}{e}$.

How do we generalize RANKING in presence of unrestricted weights b_u? The natural GREEDY algorithm which matches an arriving vertex to the highest-weighted unmatched neighbor, achieves a competitive ratio of $\frac{1}{2}$ (see Appendix B for a proof). No deterministic algorithm can do better. While the optimality of RANKING for unweighted matching suggests choosing random ranking permutations of U, RANKING itself can do as badly as factor $\frac{1}{n}$ for some weighted instances.

The main challenge in solving this problem is that a good algorithm must follow very different strategies depending on the weights in the input instance. GREEDY and RANKING are both suboptimal for this problem, but both have ideas which are essential to its solution. In particular, they perform well on distinct classes of inputs, namely, GREEDY on highly skewed weights and RANKING on equal weights. The following observation about RANKING helps us bridge the gap between these two approaches: Suppose we perturb each weight b_u identically and independently and then sort the vertices in the order of decreasing perturbed weights. When all the weights are equal, the resulting order happens to be a uniformly random permutation of U and thus, RANKING on unweighted instances can be thought of as GREEDY on perturbed weights! We use this insight to construct our solution to the vertex-weighted matching problem. While the nature of perturbation used did not matter in the above discussion, we need a very specific perturbation procedure for general vertex-weights.

Our algorithm is defined below:

Algorithm 1: PERTURBED-GREEDY

For each $u \in U$, pick a number x_u uniformly at random from $[0, 1]$.
Define the function $\psi(x) := 1 - e^{-(1-x)}$.
foreach *arriving* $v \in V$ **do**
  Match v to the unmatched neighbor $u \in U$ with the highest value of $b_u\psi(x_u)$. Break ties consistently, say by vertex id.

Remarks: It is not obvious, and indeed is remarkable in our opinion, that it suffices to perturb each weight b_u completely independently of other weights. In Appendix C, we provide intuition as to why such is the case. Also, the particular form of the function ψ is not a pre-conceived choice, but rather an artifact of our analysis. This combined with the discussion in Section 1.2 seems to suggest that ψ is the 'right' perturbation function. We note that we can also choose the function $\psi(x)$ to be $1-e^{-x}$, which keeps the algorithm and results unchanged. Finally, we note that the multipliers $y_u = \psi(x_u)$ are distributed according to the density function $f(y) = \frac{1}{1-y}$ for $y \in \left[0, 1 - \frac{1}{e}\right]$. Therefore, we could have equivalently stated our algorithm as: For each $u \in U$, choose a random multiplier $y_u \in \left[0, 1 - \frac{1}{e}\right]$ from the above distribution, and use $b_u y_u$ as the perturbed weight.

Our main result is the following theorem. The second part of the theorem follows from the optimality of RANKING for unweighted matching [KVV90].

Theorem 1 PERTURBED-GREEDY *achieves a competitive ratio of* $1 - 1/e$ *for the vertex-weighted online bipartite matching problem. No (randomized) algorithm has a better competitive ratio.*

In addition to the basic idea (from the proof of RANKING) of charging unmatched vertices in some probabilistic events to matched vertices in other events, our analysis needs to handle the new complexity introduced due to the weights on vertices. At a very high level, just like the algorithm, our analysis also manages to pull together the essence of the analyses of both GREEDY and RANKING.

1.2 Implications of the Result

Finding the optimal distribution over permutations of U: Since PERTURBED-GREEDY also chooses ranking orders through randomization, we can interpret it as a non-uniform RANKING, where it chooses permutations of U from the 'optimal' distribution. But we could have posed the following question, without the knowledge of our algorithm: How do we find an optimal non-uniform distribution over permutations of U? As a start, let us consider the case of 2×2 graphs. By exhaustive search over all 2×2 graphs, we can figure out the best RANKING like algorithm for 2×2 graphs (Figure 2 in Appendix D shows the only two potentially 'hard' instances in 2×2 graphs). This algorithm picks the permutation (u_1, u_2) with probability $\frac{\alpha}{1+\alpha}$ and the permutation (u_2, u_1) with probability $\frac{1}{1+\alpha}$ (where $\alpha = b_{u_1}/b_{u_2}$), and then proceeds to match to the highest neighbor. This algorithm gives a factor of $\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}$, which is minimized at $\alpha = 1$, giving a factor of 3/4 (in which case the algorithm is simply the same as RANKING).

An attempt to generalize this idea to larger graphs fails due to a blow-up in complexity. In general, we need a probability variable p_σ for every permutation σ of U. The expected weight of the matching produced by the algorithm on a graph G, is a linear expression $\text{ALG}_G(p_{\sigma_1}, p_{\sigma_2}, ...)$. Thus, the optimal distribution over permutations is given by the optimal solution of a linear program in the p_σ variables. But this LP has exponentially many variables (one per permutation) and constraints (one per "canonical graph instance"). Therefore, our algorithm can be thought of as solving this extremely large LP through a very simple process.

General capacities / Matching $u \in U$ multiple times: Consider the following generalization of the online vertex-weighted bipartite matching problem: Apart from a weight b_u, each vertex $u \in U$ has a capacity c_u such that u can be matched to *at most* c_u vertices in V. The capacities allow us to better model 'budgets' in many practical situations, *e.g.*, in online advertising. Our algorithm easily handles general capacities: For each $u \in U$, make c_u copies of u and solve the resulting instance with unit capacities: It is easy to verify that the solution is $\left(1 - \frac{1}{e}\right)$-approximate in expectation for the original problem with capacities.

Online budgeted allocation :- The *single bids* case vs. the *small bids* case: As noted earlier and proved in Appendix A, the special case of the online budgeted allocation problem with all the non-zero bids of an agent being equal ($b_{ij} = b_i$ or 0), reduces to our vertex-weighted matching problem. Since each agent provides a single bid value for all items, let us call this restriction the *single bids* case.

Corollary 2 PERTURBED-GREEDY *achieves a competitive ratio of* $1 - 1/e$ *for the* single bids *case of the online budgeted allocation problem.*

Note that the *small bids* case ($b_{ij} \ll B_i$) studied in [MSVV05, BJN07] does not reduce to or from the *single bids* case. Yet, as it turns out, PERTURBED-GREEDY is equivalent to the algorithm of [MSVV05] - let us call it MSVV - on instances that belong to the intersection of the two cases. When every agent has a *single small bid* value, the problem corresponds to vertex-weighted matching with large capacities c_u for every vertex u. Recall that we handle capacities on $u \in U$ by making c_u copies $u_1, u_2, ..., u_{c_u}$ of u. For each of these copies, we choose a random $x_{u_i} \in [0, 1]$ uniformly and independently. In expectation, the x_{u_i}'s are uniformly distributed in the interval $[0, 1]$. Also observe that PERTURBED-GREEDY will match $u_1, u_2, ..., u_{c_u}$ in the increasing order of x_{u_i}'s, if at all. Therefore, at any point in the algorithm, if u_i is the unmatched copy of u with smallest x_{u_i} (and consequently highest multiplier $\psi(x_{u_i})$) then x_{u_i} is in expectation equal to the fraction of the capacity c_u used up at that point. But MSVV uses exactly the scaling factor $\psi(T)$ where T is the fraction of spent budget at any point. We conclude that in expectation, PERTURBED-GREEDY tends to MSVV as the capacities grow large, in the single small bids case.

It is important to see that this phenomenon is not merely a consequence of the common choice of function ψ. In fact, the function ψ is not a matter of choice at all - it is a by-product of both analyses (Refer to the remark at the end of Section 3). The fact that it happens to be the exact same function seems to suggest that ψ is the 'right' function. Moreover, the analyses of the two algorithms do not imply one-another. Our variables are about expected gains and losses over a probability space, while the algorithm in [MSVV05] is purely deterministic.

This smooth 'interface' between the seemingly unrelated *single bids* and *small bids* cases hints towards the existence of a unified solution to the general online budgeted allocation problem.

1.3 Other Related Work

Our problem is a special case of online bipartite matching with edge weights, which has been studied extensively in the literature. With general edge weights and vertices arriving in adversarial order, every algorithm can be arbitrarily bad (see Appendix G). There are two ways to get around this hardness: (a) assume that vertices arrive in a random order, and/or (b) assume some restriction on the edge weights.

When the vertices arrive in random order, it corresponds to a generalization of the *secretary* problem to transversal matroids [BIK07]. Dimitrov and Plaxton [DP08] study a special case where the weight of an edge (u, v) depends only on the vertex v – this is similar to the problem we study, except that it assumes a random arrival model (and assumes vertex weights on the *online* side). Korula and Pal [KP09] give an $\frac{1}{8}$-competitive algorithm for the problem with general edge weights and for the general *secretary* problem on transversal matroids.

If one does not assume random arrival order, every algorithm can be arbitrarily bad with general edge weights or even with weights on arriving vertices. [KP93] introduce the assumption of edge weights coming from a metric space and give an optimal deterministic algorithm with a competitive factor of $\frac{1}{3}$. As far as we know, no better randomized algorithm is known for this problem.

Finally, there has been other recent work [DH09, GM08, FMMM09], although not directly related to our results, which study online bipartite matching and budgeted allocations in stochastic arrival settings.

Roadmap: The rest of the paper is structured as follows: In Section 2 we set up the preliminaries and provide a warm up analysis of a proof of RANKING in the unweighted special case. Section 3 contains the proof of Theorem 1.

2 Preliminaries

2.1 Problem Statement

Consider an undirected bipartite graph $G(U, V, E)$. The vertices of U, which we will refer to as the *offline* side, are known from the start. We are also given a weight b_u for each vertex $u \in U$. The vertices of V, referred to as the *online* side, arrive one at a time (in an arbitrary order). When a vertex v arrives, all the edges incident to it are revealed, and at this point, the vertex v can be matched to one of its unmatched neighbors (irrevocably) or left permanently unmatched. The goal is to maximize the sum of the weights of matched vertices in U.

Let permutation π represent the arrival order of vertices in V and let M be the subset of matched vertices of U at the end. Then for the input (G, π), the gain of the algorithm, denoted by $ALG(G, \pi)$, is $\sum_{u \in M} b_u$.

We use competitive analysis to analyze the performance of an algorithm. Let $M^*(G)$ be an optimal (offline) matching, i.e. one that maximizes the total gain for G (note that the optimal matching depends only on G, and is independent of π), and let $\text{OPT}(G)$ be the total gain achieved by $M^*(G)$. Then the competitive ratio of an algorithm is $\min_{G,\pi} \frac{\mathrm{ALG}(G,\pi)}{\mathrm{OPT}(G)}$. Our goal is to devise an online algorithm with a high competitive ratio.

Definition 1 **($M^*(G)$)** *For a given G, we will fix a particular optimal matching, and refer to it as the optimal offline matching $M^*(G)$.*

Definition 2 **(u^*)** *Given a G, its optimal offline matching $M^*(G)$ and a $u \in U$ that is matched in $M^*(G)$, we define $u^* \in V$ as its partner in $M^*(G)$.*

2.2 Warm-up: Analysis of Ranking for Unweighted Online Bipartite Matching

Recall that online bipartite matching is a special case of our problem in which the weight of each vertex is 1, i.e. $b_u = 1$ for all $u \in U$. [KVV90] gave an elegant randomized algorithm for this problem and showed that it achieves a competitive ratio of $(1-1/e)$ in expectation. In this section, we will re-prove this classical result as a warm-up for the proof of the main result. The following proof is based on those presented by [BM08, GM08] previously.

Algorithm 2: Ranking

Choose a random permutation σ of U uniformly from the space of all permutations.
foreach *arriving* $v \in V$ **do**
  Match v to the unmatched neighbor in u which appears earliest in σ.

Theorem 3 **([KVV90])** *In expectation, the competitive ratio of* Ranking *is at least* $1 - \frac{1}{e}$.

In this warm-up exercise, we will simplify the analysis by making the following assumptions: $|U| = |V| = n$ and G has a perfect matching. These two assumptions imply that OPT $= n$ and that the optimal matching $M^*(G)$ is a perfect matching.

For any permutation σ, let Ranking(σ) denote the matching produced by Ranking when the randomly chosen permutation happens to be σ. For a permutation $\sigma = (u_1, u_2, ..., u_n)$ of U, we say that a vertex $u = u_t$ has rank $\sigma(u) = t$. Consider the random variable

$$y_{\sigma,i} = \begin{cases} 1 & \text{If the vertex at rank } i \text{ in } \sigma \text{ is matched by Ranking}(\sigma). \\ 0 & \text{Otherwise} \end{cases}$$

Definition 3 **(Q_t, R_t)** *Q_t is defined as the set of all occurrences of matched vertices in the probability space.*

$$Q_t = \{ \ (\sigma, t) \ : \ y_{\sigma,t} = 1 \ \}$$

Similarly, R_t is defined as the set of all occurrences of unmatched vertices in the probability space.

$$R_t = \{ \ (\sigma, t) \ : \ y_{\sigma,t} = 0 \ \}$$

Let x_t be the probability that the vertex at rank t in σ is matched in Ranking(σ), over the random choice of permutation σ. Then, $x_t = \frac{|Q_t|}{n!}$ and $1 - x_t = \frac{|R_t|}{n!}$. The expected gain of the algorithm is $\mathrm{ALG}_{G,\pi} = \sum_t x_t$.

Definition 4 **(σ_u^i)** *For any σ, let σ_u^i be the permutation obtained by removing u from σ and inserting it back into σ at position i.*

Lemma 4 *If the vertex u at rank t in σ is unmatched by* Ranking*(σ), then for every $1 \le i \le n$, u^* is matched in* Ranking*(σ_u^i) to a vertex u' such that $\sigma_u^i(u') \le t$.*

Proof: Refer to Lemma 5 in the analysis of PERTURBED-GREEDY for the proof of a more general version of this statement. □

In other words, for every vertex that remains unmatched in some event in the probability space, there are many matched vertices in many different events in the space. In the remaining part of this section, we quantify this effect by bounding $1 - x_t$, which is the probability that the vertex at rank t in σ (chosen randomly by RANKING) is unmatched, in terms of some of the x_ts.

Definition 5 (Charging map $f(\sigma, t)$) *f is a map from bad events (where vertices remain unmatched) to good events (where vertices get matched). For each $(\sigma, t) \in R_t$,*

$$f(\sigma, t) \;=\; \{(\sigma_u^i, s) \;:\; 1 \le i \le n,\ \sigma(u) = t \text{ and } \textsc{Ranking}(\sigma_u^i) \text{ matches } u^* \text{ to } u' \text{ where } \sigma_u^i(u') = s\}$$

In other words, let u be the vertex at rank t in σ. Then $f(\sigma, t)$ contains all (σ', s), such that σ' can be obtained from σ by moving u to some position and s is the rank of the vertex to which u^*, the optimal partner of u, is matched in σ'.

For every $(\sigma, t) \in R_t$, $(\pi, s) \in f(\sigma, t)$ implies $y_{\pi,s} = 1$ for some $s \le t$. Therefore,

$$\bigcup_{(\sigma,t)\in R_t} f(\sigma, t) \;\subseteq\; \bigcup_{s \le t} Q_s$$

Claim 1 *If $(\rho, s) \in f(\sigma, t)$ and $(\rho, s) \in f(\overline{\sigma}, t)$, then $\sigma = \overline{\sigma}$.*

Proof: Let u' be the vertex in ρ at rank s. Let u^* be the vertex to which u' is matched by RANKING. Then it is clear from the definition of the map f that $\rho = \sigma_u^{\rho(u)} = \overline{\sigma}_u^{\rho(u)}$, implying $\sigma = \overline{\sigma}$. □

The claim proves that for a fixed t, the set-values $f(\sigma, t)$ are disjoint for different σ. Therefore,

$$1 - x_t \;=\; \frac{|R_t|}{n!} \;=\; \frac{1}{n} \cdot \frac{\left|\bigcup_{(\sigma,t)\in R_t} f(\sigma, t)\right|}{n!} \;\le\; \frac{1}{n} \cdot \frac{\left|\bigcup_{s \le t} Q_s\right|}{n!} \;=\; \frac{1}{n} \sum_{s \le t} \frac{|Q_s|}{n!} \;=\; \frac{\sum_{s \le t} x_s}{n}$$

Therefore, the probabilities x_t's obey the equation $1 - x_t \;\le\; \frac{1}{n} \sum_{s \le t} x_s$ for all t. Since any vertex with rank 1 in any of the random permutations will be matched, $x_1 = 1$. One can make simple arguments [KVV90, BM08, GM08] to prove that under these conditions, $\mathrm{ALG}_{G,\pi} = \sum_t x_t \ge \left(1 - \frac{1}{e}\right) n = \left(1 - \frac{1}{e}\right) OPT$, thereby proving Theorem 3.

3 Proof Of Theorem 1

In this section, we will assume that $|U| = |V| = n$ and that G has a perfect matching. In Appendix F we will show how this assumption can be removed.

Recall that our algorithm works as follows: For each $u \in U$, let $\sigma(u)$ be a number picked uniformly at random from $[0, 1]$ (and independent of other vertices) Now, when the next vertex $v \in V$ arrives, match it to the available neighbor u with the maximum value of $b_u \psi(\sigma(u))$, where $\psi(x) := 1 - e^{-(1-x)}$.

For ease of exposition, we will prove our result for a discrete version of this algorithm. For every $u \in U$ we will choose a random integer $\sigma(u)$ uniformly from $\{1, ..., k\}$ where k is the parameter of discretization. We will also replace the function $\psi(x)$ by its discrete version $\psi(i) = 1 - \left(1 - \frac{1}{k}\right)^{-(k-i+1)}$. The discrete version of our algorithm also matches each incoming vertex $v \in V$ to the available neighbor u with the maximum value of $b_u \psi(\sigma(u))$. Notice that ψ is a decreasing function, so $\psi(s) \ge \psi(t)$ if $s \le t$. As $k \to \infty$, the discrete version tends to our original algorithm.

We begin with some definitions, followed by an overview of the proof.

We will denote by $\sigma \in [k]^n$, the set of these random choices. We will say that u is at *position* t in σ if $\sigma(u) = t$. As a matter of notation, we will say that position s is *lower* (resp. higher) than t if $s \le t$ (resp. $s \ge t$).

Definition 6 (**u is matched in σ**) *We say that u is matched in σ if our algorithm matches it when the overall choice of random positions happens to be σ.*

Let $y_{\sigma,t}$ be the indicator variable denoting that the vertex at position t is matched in σ.

Definition 7 (**Q_t, R_t**) *Q_t is defined as the set of all occurrences of matched vertices in the probability space.*

$$Q_t = \{(\sigma, t, u) \ : \ \sigma(u) = t \text{ and } y_{\sigma,t} = 1\}$$

Similarly, R_t is defined as the set of all occurrences of unmatched vertices in the probability space.

$$R_t = \{(\sigma, t, u) \ : \ \sigma(u) = t \text{ and } y_{\sigma,t} = 0\}$$

Let x_t be the *expected gain* at t, over the random choice of σ. Then,

$$x_t = \frac{\sum_{(\sigma,t,u)\in Q_t} b_u}{k^n} \tag{1}$$

The expected gain of the algorithm is $\mathrm{ALG}_{G,\pi} = \sum_t x_t$. Also note that the *optimal gain* at any position t is $B = \frac{\mathrm{OPT}(G)}{k}$ since each vertex in U appears at position t with probability $1/k$ and is matched in the optimal matching. Therefore,

$$B - x_t = \frac{\sum_{(\sigma,t,u)\in R_t} b_u}{k^n} \tag{2}$$

Definition 8 (**σ_u^i**) *For any σ, $\sigma_u^i \in [k]^n$ is obtained from σ by changing the position of u to i, i.e. $\sigma_u^i(u) = i$ and $\sigma_u^i(u') = \sigma(u')$ for all $u' \neq u$.*

Observation 1 *For all $(\sigma, t, u) \in R_t$ and $1 \leq i \leq k$, our algorithm matches u^* to some $u' \in U$ in σ_u^i.*

The above observation follows from Lemma 5. We'll use it to define a map from bad events to good events as follows.

Definition 9 (**Charging Map $f(\sigma, t, u)$**) *For every $(\sigma, t, u) \in R_t$, define the set-valued map*

$$f(\sigma, t, u) = \{(\sigma_u^i, s, u') \ : \ 1 \leq i \leq k, \text{ and the algorithm matches } u^* \text{ to } u' \text{ in } \sigma_u^i \text{ where } \sigma_u^i(u') = s\}$$

Observation 2 *If $(\rho, s, u') \in f(\sigma, t, u)$, then $(\rho, s, u') \in Q_s$.*

Now we are ready to give an overview of the proof.

Overview of the proof

The key idea in the analysis of Ranking in Section 2.2 was that we can bound the number of occurrences of unmatched vertices - the *bad* events - in the entire probability space by a careful count of the matched vertices - the *good* events. The charging map f defined above is an attempt to do this. We'll show in Lemma 5 that if $(\sigma_u^i, s, u') \in f(\sigma, t, u)$, then the scaled (by ψ) gain due to u' in σ_u^i is no less than the scaled loss due to u in σ. However, s may be higher or lower than t, unlike Ranking where $s \leq t$. This implies that the bound is in terms of events in $\bigcup_s Q_s$, $1 \leq s \leq k$, which is very weak (as many of the events in the union are not used).

One idea is to bound the sum of losses incurred at all positions, thereby using almost all the events in $\bigcup_s Q_s$. However, if we do this, then the charging map loses the disjointness property, i.e. if $(\sigma, t, u) \in R_t$ and $(\sigma_u^i, i, u) \in R_i$ then f value of both these occurrences is the same. Thus, each event in $\bigcup_s Q_s$ gets charged several times (in fact a non-uniform number of times), again making the bound weak. To this end, we introduce the idea of *marginal loss* (Definition 10), which helps us define a disjoint map and get a tight bound.

Next, we formalize the above.

Formal proof

We begin by proving an analogue of Lemma 4.

Lemma 5 *If the vertex u at position t in σ is unmatched by our algorithm, then for every $1 \leq i \leq k$, the algorithm matches u^* in σ_u^i to a vertex u' such that $\psi(t)b_u \leq \psi\left(\sigma_u^i(u')\right) b_{u'}$.*

Proof: **Case 1 ($i \geq t$)**: Let $v_1, ..., v_n$ be the order of arrival of vertices in V. Clearly, v_1 will see the same choice of neighbors in σ_u^i as in σ, except the fact that the position of u is higher in σ_u^i than in σ. Since we did not match v_1 to u in σ, v_1 will retain its match from σ even in σ_u^i. Now assuming that $v_1, ..., v_l$ all match the same vertex in σ_u^i as they did in σ, v_{l+1} will see the same choice of neighbors in σ_u^i as in σ with the exception of u. Since v_{l+1} did not match u in σ either, it will retain the same neighbor in σ_u^i and by induction every vertex from V, specifically u^* keeps the same match in σ_u^i as in σ. Since $\sigma(u') = \sigma_u^i(u')$, we conclude $\psi(t)b_u \leq \psi\left(\sigma_u^i(u')\right) b_{u'}$.

Case 2 ($i < t$): For a vertex $v \in V$, let $m_\sigma(v)$ and $m_{\sigma_u^i}(v)$ be the vertices to which v is matched in σ and σ_u^i respectively, if such a match exists and null otherwise. Intuitively, since $\psi(i) \geq \psi(t)$, the scaling factor of b_u only improves in this case, while that of any other vertex in U remains the same. Therefore, we can expect u to be more likely to be matched in σ_u^i and the $\psi\left(\sigma_u^i\left(m_{\sigma_u^i}(v)\right)\right) b_{m_{\sigma_u^i}(v)} \geq \psi\left(\sigma\left(m_\sigma(v)\right)\right) b_{m_\sigma(v)}$ to hold for all $v \in V$. In fact, something more specific is true. The symmetric difference of the two matchings produced by the algorithm for σ and σ_u^i is exactly one path starting at u that looks like $(u,\ v_1,\ m_\sigma(v_1),\ v_2,\ m_\sigma(v_2),\ ...)$, where $(v_1, v_2, ...)$ appear in their order of arrival. In what follows we prove this formally.

Let $V' = \{v \in V\ :\ m_\sigma(v) \neq m_{\sigma_u^i}(v)\}$ be the set of vertices in V with different matches in σ and σ_u^i. Index the members of V' as $v_1, ..., v_l$ in the same order as their arrival, *i.e.* v_1 arrives the earliest. For simplicity, let $u_j = m_\sigma(v_j)$ and $w_j = m_{\sigma_u^i}(v_j)$.

We assert that the following invariant holds for $2 \leq j \leq l$: Both u_j and u_{j-1} are unmatched in σ_u^i when v_j arrives and v_j matches u_{j-1}, *i.e.* $w_j = u_{j-1}$.

For base case, observe that the choice of neighbors for v_1 in σ_u^i is the same as in σ, except u, which has moved to a lower position. Since by definition v_1 does not match u_1 in σ_u^i, $w_1 = u$. Now consider the situation when v_2 arrives. All the vertices arriving before v_2 - with the exception of v_1 - have been matched to the same vertex in σ_u^i as in σ, and v_1 has matched to u, leaving u_1 yet unmatched. Let $U_\sigma(v_2)$ and $U_{\sigma_u^i}(v_2)$ be the sets of unmatched neighbors of v_2 in σ and σ_u^i respectively *at the moment* when v_2 arrives. Then from above arguments, $U_{\sigma_u^i}(v_2) = (U_\sigma(v_2) \cup \{u_1\}) - \{u\}$. Since u was unmatched in σ, $u_2 \neq u$. Since $v_2 \in V'$, $w_2 \neq u_2$. This is only possible if $w_2 = u_1$. And hence the base case is true.

Now assume that the statement holds for $j-1$ and consider the arrival of v_j. By induction hypothesis, v_1 has been matched to u and $v_2, .., v_{j-1}$ have been matched to $u_1, ..., u_{j-2}$ respectively. All the other vertices arriving before v_j that are not in V' have been matched to the same vertex in σ_u^i as in σ. Therefore, u_{j-1} is yet unmatched. Let $U_\sigma(v_j)$ and $U_{\sigma_u^i}(v_j)$ be the sets of unmatched neighbors of v_j in σ and σ_u^i respectively at the moment when v_j arrives. Then from above arguments, $U_{\sigma_u^i}(v_j) = (U_\sigma(v_j) \cup \{u_{j-1}\}) - \{u\}$. Since u was unmatched in σ, $u_j \neq u$. Given that $w_j \neq u_j$, the only possibility is $w_j = u_{j-1}$. Hence the proof of the inductive statement is complete.

If $u^* \notin V'$ then $u' = m_{\sigma_u^i}(u^*) = m_\sigma(u^*)$ and the statement of the lemma clearly holds since $\sigma(u') = \sigma_u^i(u')$. If $u^* = v_1$, then $u' = u$ and $\psi\left(\sigma_u^i(u')\right) b_{u'} = \psi(i)b_u \geq \psi(t)b_u$ since $i < t$. Now suppose $u^* = v_j$ for some $j \geq 2$. Then $u' = u_{j-1}$ and by the invariant above,

$$\begin{aligned}
\psi\left(\sigma_u^i(u')\right) b_{u'} \ =\ \psi\left(\sigma_u^i(u_{j-1})\right) b_{u_{j-1}} &\geq \psi\left(\sigma_u^i(u_j)\right) b_{u_j} && (3)\\
&= \psi\left(\sigma(u_j)\right) b_{u_j} && (4)\\
&\geq \psi(t) b_u && (5)
\end{aligned}$$

Equation (3) follows from the fact that $u^* = v_j$ was matched in σ_u^i to u_{j-1} when u_j was also unmatched. The fact that only u changes its position between σ and σ_u^i leads us to (4). Finally, equation (5) follows from the fact that u^* was matched to u_j in σ when u was also unmatched. $\square$

Using the above lemma, we get the following easy observation.

Observation 3 *For all $(\sigma, t, u) \in R_t$, $1 \le t \le k$, $f(\sigma, t, u)$ contains k values.*

Remark: As noted in the overview, although Lemma 5 looks very similar to Lemma 4, it is not sufficient to get the result, since the good events pointed to by Lemma 5 are scattered among all positions $1 \le s \le k$ – in contrast to Lemma 4, which pointed to only lower positions $s \le t$, giving too weak a bound. We try to fix this by combining the losses from all R_t. However we run into another difficulty in doing so. While for any fixed t, the maps $f(\sigma, t, u)$ are disjoint for all $(\sigma, t, u) \in R_t$, but the maps for two occurrences in different R_ts may not be disjoint. In fact, whenever some u is unmatched in σ at position t, it will also remain unmatched in σ_u^j for $j > t$, and the sets $f(\sigma, t, u)$ and $f(\sigma_u^j, j, u)$ will be exactly the same! This situation is depicted in Figure 3 in Appendix E.

This absence of disjointness again renders the bound too weak. To fix this, we carefully select a subset of bad events from $\bigcup_t R_t$ such that their set-functions are indeed disjoint, while at the same time, the total gain/loss can be easily expressed in terms of the bad events in this subset.

Definition 10 (Marginal loss events S_t) *Let $S_t = \{(\sigma, t, u) \in R_t : (\sigma_u^{t-1}, t-1, u) \notin R_{t-1}\}$, where $R_0 = \emptyset$.*

Informally, S_t consists of *marginal* losses. If u is unmatched at position t in σ, but matched at position $t-1$ in σ_u^{t-1}, then $(\sigma, t, u) \in S_t$ (See Figure 3 in Appendix E). The following property can be proved using the same arguments as in Case 1 in the proof of Lemma 5.

Observation 4 *For $(\sigma, t, u) \in S_t$, u is matched at i in σ_u^i if and only if $i < t$.*

Definition 11 (Expected Marginal Loss α_t)

$$\textit{Expected marginal loss at position } t \;=\; \alpha_t \;=\; \frac{\sum_{(\sigma,t,u)\in S_t} b_u}{k^n} \tag{6}$$

Claim 2

$$\forall\ t: \qquad x_t \;=\; B - \sum_{s \le t} \alpha_s \tag{7}$$

$$\textit{Total loss} \;=\; \sum_t (B - x_t) \;=\; \sum_t (k - t + 1)\alpha_t \tag{8}$$

Proof: To prove equation (7), we will fix a t and construct a one-to-one map $g : R_t \to \bigcup_{s \le t} S_t$. Given $(\sigma, t, u) \in R_t$, let i be the lowest position of u such that u remains unmatched in σ_u^i. By observation 4, i is unique for (σ, t, u). We let $g(\sigma, t, u) = (\sigma_u^i, i, u)$. Clearly, $(\sigma_u^i, i, u) \in S_i$. To prove that the map is one-to-one, suppose $(\rho, s, u) = g(\sigma, t, u) = g(\overline{\sigma}, t, u)$. Then by definition of g, $\rho = \sigma_u^s = \overline{\sigma}_u^s$ which is only possible if $\sigma = \overline{\sigma}$. Therefore, $|R_t| = \bigcup_{s \le t} S_t$.

Lastly, observe that g maps an element of R_t corresponding to the vertex u being unmatched, to an element of S_i corresponding to the same vertex u being unmatched. From equation (2),

$$B - x_t \;=\; \frac{\sum_{(\sigma,t,u)\in R_t} b_u}{k^n} \;=\; \sum_{i \le t} \frac{\sum_{(\sigma_u^i,i,u)\in S_i} b_u}{k^n} \;=\; \sum_{i \le t} \alpha_i$$

This proves equation (7). Summing (7) for all t, we get (8). $\square$

Now consider the same set-valued map f from Definition 9, but restricted only to the members of $\bigcup_t S_t$. We have:

Claim 3 *For $(\sigma, t, u) \in S_t$ and $(\overline{\sigma}, \overline{t}, \overline{u}) \in S_{\overline{t}}$, if $(\rho, s, u') \in f(\sigma, t, u)$ and $(\rho, s, u') \in f(\overline{\sigma}, \overline{t}, \overline{u})$ then $\sigma = \overline{\sigma}$, $t = \overline{t}$ and $u = \overline{u}$.*

Proof: If u' is matched to v in ρ then by definition of f, $v = u^* = \overline{u}^*$, implying $u = \overline{u}$. Therefore, $\rho = \sigma_u^i = \overline{\sigma}_u^i$ for some i. But this implies that $\overline{\sigma} = \sigma_u^j$ for some j. This is only possible for $j = t$ since by definition, if u is unmatched in σ at t, then there exists a unique i for which $(\sigma_u^i, i, u) \in \bigcup_t S_t$. If $j = t$, then $\sigma = \overline{\sigma}$ and $t = \overline{t}$. $\square$

Armed with this disjointness property, we can now prove our main theorem.

Theorem 6 *As $k \to \infty$,*

$$\sum_t x_t \ \geq \ \left(1 - \frac{1}{e}\right) \mathrm{OPT}(G) \tag{9}$$

Proof: Using Lemma 5 and Observation 3, we have for every $(\sigma, t, u) \in S_t$,

$$\psi(t) b_u \ \leq \ \frac{1}{k} \sum_{(\sigma_u^i, s, u') \in f(\sigma, t, u)} \psi(s) b_{u'} \tag{10}$$

If we add the equation (10) for all $(\sigma, t, u) \in S_t$ and for all $1 \leq t \leq n$, then using Claim 3 and Observation 2, we arrive at

$$\begin{aligned}
\sum_t \psi(t) \frac{\sum_{(\sigma,t,u) \in S_t} b_u}{k^n} \ &\leq \ \frac{1}{k} \sum_t \psi(t) \frac{\sum_{(\sigma,t,u) \in Q_t} b_u}{k^n} \\
\sum_t \psi(t) \alpha_t \ &\leq \ \frac{1}{k} \sum_t \psi(t) x_t \qquad (11) \\
&= \ \frac{1}{k} \sum_t \psi(t) \left(B - \sum_{s \leq t} \alpha_s\right) \qquad (12)
\end{aligned}$$

Equation (11) follows from (6) and (1). Equation (12) uses Claim 2.

We now rearrange terms to get

$$\sum_t \alpha_t \left(\psi(t) + \frac{\sum_{s \geq t} \psi(s)}{k}\right) \ \leq \ \frac{B}{k} \sum_t \psi(t) \tag{13}$$

When $\psi(t) = 1 - \left(1 - \frac{1}{k}\right)^{k-t+1}$, observe that $\psi(t) + \frac{\sum_{s \geq t} \psi(s)}{k} \geq \frac{(k-t+1)}{k}$ and $\sum_t \psi(t) = \frac{k}{e}$ as $k \to \infty$. Using Claim 2,

$$\begin{aligned}
\text{Total loss} \ &= \ \sum_t (B - x_t) \ = \ \sum_t (k - t + 1) \alpha_t \\
&\leq \ k \sum_t \alpha_t \left(\psi(t) + \frac{\sum_{s \geq t} \psi(s)}{k}\right) \\
&\leq \ B \sum_t \psi(t) \\
&= \ \frac{kB}{e} \qquad \text{as } k \to \infty \\
&= \ \frac{\mathrm{OPT}(G)}{e}
\end{aligned}$$

Hence, as $k \to \infty$,

$$\text{Total gain} \ \geq \ \left(1 - \frac{1}{e}\right) \mathrm{OPT}(G)$$

Remark: Observe that we substituted for $\psi(t)$ only after equation (13) - up until that point, any choice of a non-increasing function ψ would have carried the analysis through. In fact, the chosen form of ψ is a result of trying to reduce the left hand side of equation (13) to the expected total loss. To conclude, the 'right' perturbation function is dictated by the analysis and not vice versa. □

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A The Reduction from Online Budgeted Allocation with Single Bids

In this section, we will show that the *single bids* case of the online budgeted allocation problem reduces to online vertex-weighted bipartite matching. Let us first define these problems.

Online budgeted allocation: We have n agents and m items. Each agent i specifies a monetary budget B_i and a bid b_{ij} for each item j. Items arrive online, and must be immediately allocated to an agent. If a set S of items is allocated to agent i, then the agent pays the minimum of B_i and $\sum_{j \in S} b_{ij}$. The objective is to maximize the total revenue of the algorithm.

Single bids case: Any bid made by agent i can take only two values: b_i or 0. In other words, all the non-zero bids of an agent are equal.

Claim 4 *Online budgeted allocation with single bids reduces to online vertex-weighted bipartite matching.*

Proof: Given an instance of online budgeted allocation where agent i has budget B_i and single bid value b_i, we will construct an input instance $G(U, V, E, \{b_u\}_{u \in U})$ of online vertex-weighted bipartite matching. The set V consists of one vertex corresponding to every item. The set U will contain one or more vertices for every agent.

For every agent i, let n_i be the largest integer such that $n_i b_i \leq B_i$ and let $r_i = B_i - n_i b_i$. Clearly, $r_i < b_i$. We will construct a set U_i of n_i vertices, each with weight b_i. In addition, if $r_i > 0$, then we will construct a vertex $\bar{u}_i$ with weight r_i and add it to U_i. For all $u \in U_i$ and $v \in V$, the edge $uv \in E$ if and only if agent i makes a non-zero bid on the item corresponding to v.

(1) Given a solution to the budgeted allocation problem where a set S_i of items is allocated to agent i, let us see how to construct a solution to the vertex-weighted matching problem with the same total value.

- If agent i pays a total of $|S_i| \cdot b_i$, then we know that $|S_i| \leq n_i$. Hence, for every item in S_i, we will match the corresponding vertex in V to a vertex in $U_i - \{\bar{u}_i\}$. Let R_i be the set of vertices in U_i thus matched. We have:

$$\sum_{u \in R_i} b_u \;=\; |R_i| \cdot b_i \;=\; |S_i| \cdot b_i$$

- If agent i pays a total amount strictly less than $|S_i| \cdot b_i$, then we know that: (a) $|S_i| \geq n_i + 1$, (b) $r_i > 0$ and (3) agent i pays the budget B_i. We can now choose any $n_i + 1$ items in S_i and match the corresponding vertices in V to the $n_i + 1$ vertices in U_i. The sum of the weights of matched vertices in U_i, $\sum_{u \in U_i} b_u = B_i$.

Summing over all i, the weight of the matching formed is equal to the total revenue of the budgeted allocation. Let OPT_A and OPT_M denote the values of the optimal solutions of the budgeted allocation and the vertex-weighted matching problems respectively. Then we conclude from the above discussion that:

$$\mathrm{OPT}_M \;\geq \mathrm{OPT}_A \qquad (14)$$

(2) Given a solution to the vertex-weighted matching problem where a set $R \subseteq U$ of vertices is matched, let us see how to construct a solution to the budgeted allocation problem with at least the same total value. Let $R_i = R \cap U_i$. For every $v \in V$ that is matched to a vertex in R_i, we will allocate the corresponding item to agent i. Let S_i be the set of items allocated to agent i.

- If $|R_i| = |S_i| \leq n_i$, then agent i pays a total of $|S_i| \cdot b_i$ and we have:

$$\sum_{u \in R_i} b_u \;\leq\; |R_i| \cdot b_i \;=\; |S_i| \cdot b_i$$

- If on the other hand, $|R_i| = |S_i| = n_i + 1$ then agent i pays a total of B_i and we have:

$$\sum_{u \in R_i} b_u \;=\; \sum_{u \in U_i} b_u \;=\; B_i$$

Summing over all i, the total revenue of the budgeted allocation is at least the weight of the matching. Let ALG_M be the expected weight of the vertex-weighted matching constructed by Perturbed-Greedy and ALG_A be the expected value of the budgeted allocation constructed using the above scheme. From the above discussion, we conclude: Therefore,

$$\begin{aligned}
\mathrm{ALG}_A &\geq \mathrm{ALG}_M \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}_M \qquad (15) \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}_A
\end{aligned}$$

Here, equation (15) follows from the main result - Theorem 1 - and the last step uses equation (14). This completes our proof.

□

B Performance of Greedy and Ranking

With non-equal weights, it is clearly preferable to match vertices with larger weight. This leads to the following natural algorithm.

Algorithm 3: Greedy

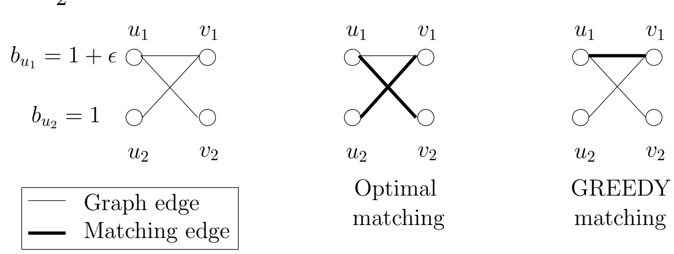
foreach *arriving* $v \in V$ **do**
  Match v to the unmatched neighbor in u which maximizes b_u (breaking ties arbitrarily);

It is not hard to show that Greedy achieves a competitive ratio of at least $\frac{1}{2}$.

Lemma 7 Greedy *achieves a competitive ratio of 1/2 in vertex-weighted online bipartite matching.*

Proof: Consider an optimal offline matching, and a vertex $u \in U$ that is matched in the optimal offline matching but not in the greedy algorithm. Now look at a vertex $u^* \in V$ that is matched to the vertex u in the optimal matching. In Greedy, u^* must have been matched to a vertex $u^{'} \in U$, s.t. $b_u \leq b_{u^{'}}$, since u was unmatched when u^* was being matched. So we'll charge the loss of b_u to $u^{'}$. Note that each $u^{'}$ does not get charged more than once – it is charged only by the optimal partner of its partner in the algorithm's matching. Thus the loss of the algorithm is no more than the value of the matching output by the algorithm. Hence the claim. □

In fact, this factor 1/2 is tight for Greedy as shown by an instance consisting of many copies of the following gadget on four vertices, with $u_1, u_2 \in U$ and $v_1, v_2 \in V$. As $\epsilon \to 0$, the competitive ratio of Greedy tends to $\frac{1}{2}$.



Notice that this counter-example relies on weights being roughly equal. We, however, know that Ranking has an expected competitive ratio of $(1 - 1/e)$ when the weights are equal. On the other hand, if the weights are very different, i.e. ϵ is large, in the above example, then Greedy provides a good competitive ratio. At the same time, if we exchanged the weights on the two vertices in the example to be $b_{u_1} = 1$ and $b_{u_2} = 1 + \epsilon$, then as ϵ grows large, the expected competitive ratio of Ranking drops to $\frac{1}{2}$ and on larger examples, it can be as low as $\frac{1}{n}$. To summarize, Greedy tends to perform well when the weights are highly skewed and Ranking performs well when the weights are roughly equal.

C Intuition Behind the Sufficiency of Independent Perturbations

Recall that our algorithm perturbs each weight b_u independent of the other weights. The fact that Perturbed-Greedy achieves the best possible competitive ratio is a post-facto proof that such independence in perturbations is sufficient. Without the knowledge of our algorithm, one could reasonably believe that the vector of vertex-weights $\{b_u\}_{u \in U}$ - which is known offline - contains valuable information which can be exploited. In what follows we provide intuition as to why this is not the case.

Consider the two input instances in Figure 1. Both the connected components in G_1 have equal weights, and hence we know that Ranking achieves the best possible competitive ratio on G_1. Similarly, both connected components in G_2 have highly skewed weights, suggesting Greedy as the optimal algorithm. On the other hand, Ranking and Greedy are far from optimal on G_2 and G_1 respectively. Since two instances with identical values of vertex-weights require widely differing strategies, this exercise suggests that we may not be losing must information by perturbing weights independently. The optimality of our algorithm proves this suggestion.

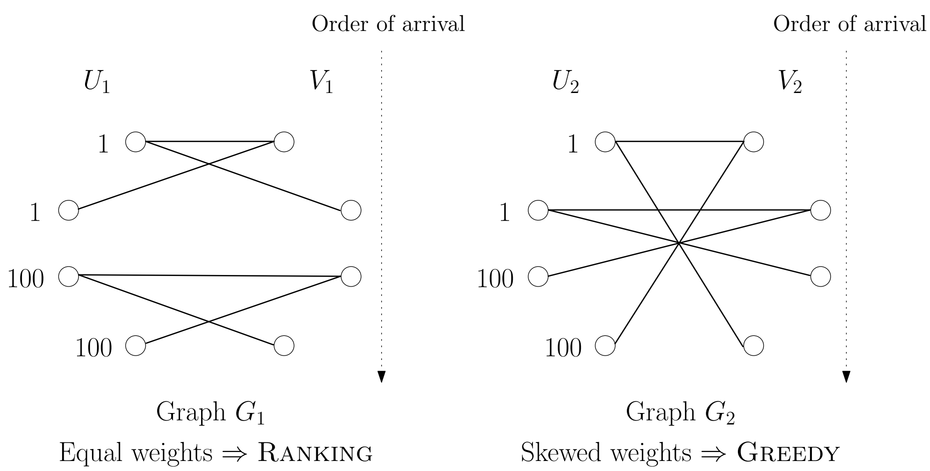


Figure 1: Two instances with the same vertex-weights, but widely differing optimal strategies.

D Hard Instances in 2×2 Graphs

Figure 2 shows the only two potentially 'hard' instances in 2×2 graphs. On all other instances, the optimal matching is found by any reasonable algorithm that leaves a vertex $v \in V$ unmatched only if all its neighbors are already matched.

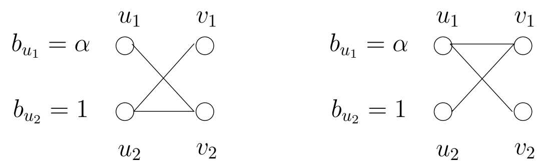


Figure 2: Canonical examples for 2×2 graphs.

E Marginal Loss Events

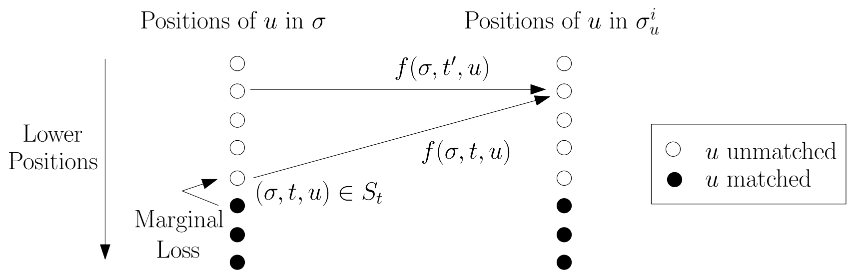


Figure 3: Marginal Losses

F Graphs with Imperfect Matchings

In Section 3, we proved Theorem 1 for graphs $G(U, V, E)$ such that $|U| = |V|$ and G has a perfect matching. We can remove these assumptions with just a few modifications to the definitions and equations involved in the proof. The algorithm remains unchanged, *i.e.* we just use PERTURBED-GREEDY. We will only outline these modifications and the rest of the proof follows easily. Let $M^*(G)$ be a maximum weight matching in $G(U, V, E)$ and $\overline{U}$ be the set of vertices in U matched by $M^*(G)$. Thus we know that $\text{OPT}(G) = \sum_{u \in \overline{U}} b_u$.

Keeping the definition of Q_t the same, we change the definition of R_t to:

$$R_t = \{(\sigma, t, u) \ : \ u \in \overline{U} \text{ and } \sigma(u) = t \text{ and } y_{\sigma,t} = 0\}$$

The above redefinition conveys the fact that if a vertex u is *not* matched by $M^*(G)$, then we no longer consider u being unmatched a bad event. Consequently, equation (2) changes to:

$$B - x_t \leq \frac{\sum_{(\sigma,t,u) \in R_t} b_u}{k^n}$$

which in turn yields following counterpart of equation (7):

$$\forall t, \qquad x_t \ \geq \ B - \sum_{s \leq t} \alpha_s \tag{16}$$

Let $\text{Eq}(t)$ be the version of (16) for t. We then multiply $\text{Eq}(t)$ by $\psi(t) - \psi(t+1)$ and sum over $1 \leq t \leq n$ to obtain a combined inequality (with $\psi(k+1) = 0$):

$$\sum_t (\psi(t) \ - \ \psi(t+1))\, x_t \ \ \geq \ \ \psi(1)B \ - \ \sum_t \psi(t)\alpha_t$$

$$\sum_t \psi(t)\alpha_t \ \ \geq \ \ \psi(1)\frac{\text{OPT}(G)}{k} \ - \ \sum_t \frac{(1 - \psi(t+1))}{k} x_t \tag{17}$$

Equation (17) used the definition of $\psi(t) = 1 - \left(1 - \frac{1}{k}\right)^{(k-t+1)}$. Combining equation (17) with (11), we get:

$$\begin{aligned}
\frac{1}{k}\sum_t \psi(t)x_t &\geq \psi(1)\frac{\text{OPT}(G)}{k} - \sum_t \frac{(1-\psi(t+1))}{k}x_t \\
\sum_t x_t &\geq \psi(1)\text{OPT}(G) - \sum_t \left(\psi(t) - \psi(t+1)\right) x_t \\
&\geq \left(1 - \frac{1}{e}\right)\text{OPT}(G)
\end{aligned}$$

as $k \to \infty$, since $\psi(1) \to \left(1 - \frac{1}{e}\right)$ and $\psi(t) - \psi(t+1) = \frac{(1-\psi(t+1)}{k} \to 0$ as $k \to \infty$.

G A Lower Bound for Randomized Algorithms with Edge Weights

In this section, we will sketch the proof of a lower bound for the competitive ratio of a randomized algorithm, when the graph $G(U,V,E)$ has edge weights and our objective is to find a matching in G with maximum total weight of edges. Previous studies of this problem have only mentioned that no constant factor can be achieved when the vertices in V arrive in an online manner. However, we have not been able to find a proof of this lower bound for randomized algorithms in any literature. We prove the result when the algorithm is restricted to be scale-free. A scale-free algorithm in this context produces the exact same matching when all the edge weights are multiplied by the same factor.

Consider a graph $G(U,V,E)$ such that U contains just one vertex u and each vertex in $v \in V$ has an edge to u of weight b_v. Fix $v_1, v_2, ...$ to be the order in which the vertices of V arrive online. By Yao's principle, it suffices for us to produce a probability distribution over $b_{v_1}, b_{v_2}, ...$ such that no deterministic algorithm can perform well in expectation. We will denote the vector of edge weights in the same order in which the corresponding vertices in V arrive, *i.e.* $(b_{v_1}, b_{v_2}, ...)$ and so on. Consider the following n vectors of edge weights: For every $1 \leq i \leq n$, $\mathbf{b}_i = (D^i, D^{i+1}, ..., D^n, 0, 0, ...)$ and so on, where D is a sufficiently large number. Suppose our input distribution chooses each one of these n vectors of edge weights with equal probability.

Clearly, regardless of the vector which is chosen, $\text{OPT}(G) = D^n$. Since an algorithm is assumed to be scale-free and online, it makes the exact same decisions after the arrival of first k vertices for each of the edge weight vectors $\mathbf{b}_j$, $1 \leq j \leq k$. Therefore, it cannot distinguish between $\mathbf{b}_1, ..., \mathbf{b}_k$ after just k steps. Hence, we can characterize any algorithm by the unique k such that it matches the k'th vertex in V with a positive weight edge.

Let ALG be any deterministic algorithm that matches the k'th incoming vertex with a positive weight edge to u. Then the expected weight of the edge chosen by ALG is $\displaystyle\frac{1}{n}\sum_{i>k} D^i$. Since D is large, this is at most $\frac{c}{n}\text{OPT}(G)$, where c is some constant. Applying Yao's principle, we conclude that the competitive ratio of the best scale-free randomized algorithm for online bipartite matching with edge weights is $O\left(\frac{1}{n}\right)$.